Edge ideals of multi-whisker graphs

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第 10 回 広島岡山代数 + ゲームシンポジウム

based on a recent work jointly with M. R. Pournaki and Naoki Terai

Outline

- We define multi-whisker graphs G as a generalization of whisker graphs.
- We obtain that G is vertex-decomposable and hence sequentially Cohen-Macaulay.
- We analyze the structure of local cohomology of $S/I(G)$.
- We give formulas of the depth, the regularity and the arithmetical rank of $S/I(G)$.

1. Preliminaries

Let $G = (V(G), E(G))$ be a finite graph, where

- $V(G) = \{x_1, \ldots, x_n\}$ the set of vertices of G
- $E(G) \subset \{\{x_i, x_j\} \subset V(G) \ : i \neq j\}$ the set of edges of G

Fix an arbitrary field $\mathbb{k}$.

Definition (R. H. Villarreal, 1990)

For a graph G,

$$I(G) := (x_i x_j \ : \{x_i, x_j\} \in E(G)) \subset \mathbb{k}[x_1, \ldots, x_n] =: \mathbb{k}[V(G)]$$

is called the *edge ideal* of G.

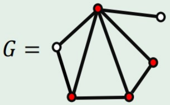
Then there is the following correspondence:

$$\left\{ \begin{array}{c} \text{EDGE IDEALS} \\ \text{(quadratic squarefree} \\ \text{monomial ideals)} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{FINITE} \\ \text{SIMPLE} \\ \text{GRAPHS} \end{array} \right\}$$

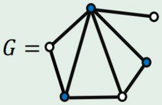
Let us recall several definitions and basic properties of combinatorial commutative algebra.

- $C \subset V(G)$ is a *vertex cover* of G
  $\overset{def}{\Longleftrightarrow} \forall \{x_i, x_j\} \in E(G),\ x_i \in C$ or $x_j \in C$
- $C \subset V(G)$ is a *minimal vertex cover* of G
  $\overset{def}{\Longleftrightarrow}$ C is a vertex cover and
  $\forall C' \subsetneq C$, C' is not a vertex cover

Example



vertex cover of G



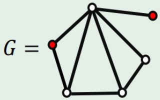
minimal vertex cover of G

- $A \subset V(G)$ is an *independent set* of G

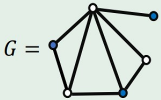
  $\overset{def}{\Longleftrightarrow} \forall x_i, \forall x_j \in A,\ \{x_i, x_j\} \notin E(G)$
- $A \subset V(G)$ is an *maximal independent set* of G

  $\overset{def}{\Longleftrightarrow}$ A is an independent set and
  $A \subsetneq \forall A' \subset V(G)$, A' is not an independent set

Example



independent set of G



maximal independent set of G

Notice that
C is a minimal vertex cover of $G \overset{iff}{\Longleftrightarrow} V(G)\backslash C$ is a maximal independent set of G.

- G is *Cohen-Macaulay* $\overset{\text{def}}{\Longleftrightarrow} \Bbbk[V(G)]/I(G)$ is Cohen-Macaulay
- G is *well-covered* $\overset{\text{def}}{\Longleftrightarrow} \forall C, \forall C'$ minimal vertex covers, $|C| = |C'|$

Let us recall the definition of whisker graphs which was introduced by Villarreal.

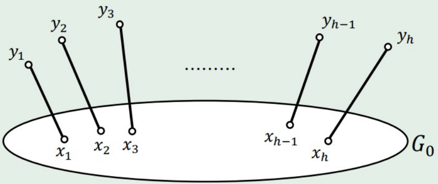
Definition (R. H. Villarreal, 1990)

Let G_0 be a graph on $V(G_0) = X_{[h]} := \{x_1, \ldots, x_h\}$.
Then, the graph $H := G_0[1, \ldots, 1]$ defined as follows:

- $V(H) := X_{[h]} \cup Y_{[h]}$, where $Y_{[h]} = \{y_1, \ldots, y_h\}$
- $E(H) := E(G_0) \cup \{\{x_1, y_1\}, \ldots, \{x_h, y_h\}\}$

H is called the *whisker graph* associated with G_0.

Image of whisker graphs

$G_0[1, \ldots, 1] =$ 

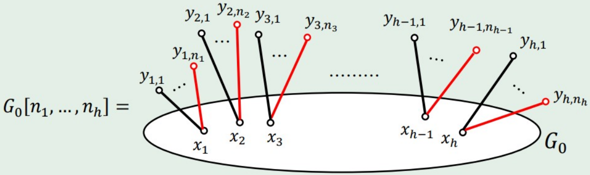
We define multi-whisker graphs as follows:

Definition (M - M. R. Pournaki - N. Terai, 2025)

Let G_0 be a graph on the vertex set $X_{[h]} = \{x_1, \ldots, x_h\}$, $n_1, \ldots, n_h \in \mathbb{Z}_{\geqq 1}$
We define the *multi-whisker graph* $G = G_0[n_1, \ldots, n_h]$ as follows:

- $V(G) := X_{[h]} \cup Y$, where $Y = \{y_{1,1}, \ldots, y_{1,n_1}\} \cup \cdots \cup \{y_{h,1}, \ldots, y_{h,n_h}\}$
- $E(G) := E(G_0) \cup \{\{x_1, y_{1,1}\}, \cdots, \{x_1, y_{1,n_1}\}, \cdots, \{x_h, y_{h,1}\}, \cdots, \{x_h, y_{h,n_h}\}\}$

Image of multi-whisker graphs



Proposition (R. H. Villarreal, 1990)

whisker graphs $\Rightarrow$ Cohen–Macaulay

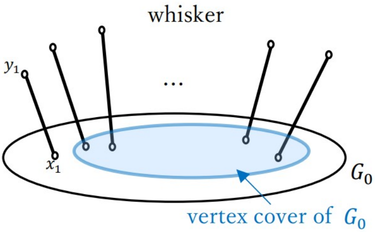
Proposition

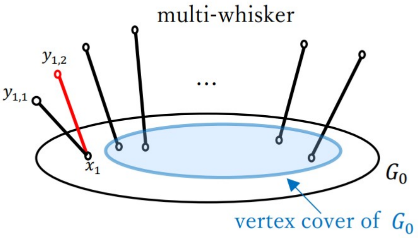
Let $G = G_0[n_1, \ldots, n_h]$ be the multi-whisker graph. TFAE:

- $n_1 = \cdots = n_h = 1$ (i.e. G is the whisker graph),
- G is Cohen-Macaulay,
- G is well-covered.

Indeed,

$$\forall n_i = 1 \Rightarrow G \text{ is Cohen-Macaulay} \Rightarrow G \text{ is well-covered} \Rightarrow \forall n_i = 1$$





3. vertex decomposablity of edge ideals of multi-whisker graphs

In commutative ring theory, the following is one of the important questions:

Question

For a ring R, under what condition is R Cohen–Macaulay?

Recall the definition of sequentially Cohen-Macaulay.

Definition (R. P. Stanley, 1996)

Let $S := \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$, M be a graded S-module.

Then, M is *sequentially Cohen-Macaulay*

$\overset{\text{def}}{\Longleftrightarrow} \exists\ 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ a filtration of M

s.t. (1) $\dim M_i/M_{i-1} < \dim M_{i+1}/M_i \quad (\forall\ i)$ and

(2) M_i/M_{i-1} is Cohen-Macaulay $\quad (\forall\ i)$

Definiton

Let G be a graph. Then,

G is *sequentially Cohen-Macaulay*

$\overset{\text{def}}{\Longleftrightarrow} \Bbbk[V(G)]/I(G)$ is sequentially Cohen-Macaulay

Let Δ be a simplicial complex.

- $\mathrm{link}_\Delta(F) := \{F' \in \Delta \mid F' \cup F \in \Delta \text{ and } F' \cap F = \phi\}$ the *link* of a face $F \in \Delta$
- $\mathrm{del}_\Delta(F) := \{F' \in \Delta \mid F' \cap F = \phi\}$ the *deletion* of a face $F \in \Delta$

Let $\Delta(G) := \{F \mid F \text{ is an independent set of G}\}$ be the independence complex of G

It is known that $I(G) = I_{\Delta(G)}$.

Definition (A. Björner - M. L. Wachs, 1997)

Let Δ be a simplicial complex on $V := \{x_1, \dots, x_n\}$, G be a graph.

- Δ is *vertex decomposable*

$\overset{\text{def}}{\iff}$ (1)$\Delta = \langle\{x_1, \dots, x_n\}\rangle$ or $\Delta = \phi$,

or

(2)(a)$\exists x \in V$ s.t. $\mathrm{link}_\Delta(\{\mathrm{x}\})$, $\mathrm{del}_\Delta(\{\mathrm{x}\})$ are vertex decomposable and

(b)$\forall F \in \mathcal{F}(\mathrm{del}_\Delta(\{\mathrm{x}\}))$, $F \in \mathcal{F}(\Delta)$

- G is *vertex decomposable*

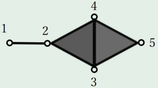
$\overset{\text{def}}{\iff}$ $\Delta(G)$ is vertex decomposable

Fact (A. Björner - M. L. Wachs, 1997 and R. P. Stanley, 1996)

Δ is vertex decomposable $\Rightarrow$ Δ is shellable $\Rightarrow$ S/I_Δ is sequentially Cohen-Macauly

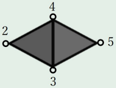
Example

Let $\Delta := \langle \{1,2\}, \{2,3,4\}, \{3,4,5\} \rangle$.

$\Delta =$ 

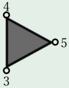
Now $\Delta \neq \langle \{1,2,3,4,5\} \rangle, \ \phi$. Also, we have

link(1)= 

del(1)= 

Check that $\mathrm{del}_\Delta(1)$ is vertex decomposable.

link(2)= 

del(2)= 

Since $\mathrm{link}_{\mathrm{del}(1)}(2) = \langle \{3,4\} \rangle$ and $\mathrm{del}_{\mathrm{del}(1)}(2) = \langle \{3,4,5\} \rangle$,
Δ is vertex decomposable.

Stating our first main result.

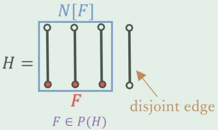
Theorem (M - M.R. Pournaki - N. Terai, 2025)

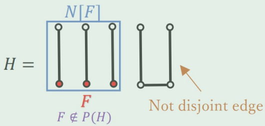
Let $G = G_0[n_1, \ldots, n_h]$ be the multi-whisker graph. Then,
G is vertex decomposable and sequentially Cohen-Macaulay.

4. Local cohomology modules of multi-whisker graphs

We define $P(H) := \{F \in \Delta(H) \mid H - \mathrm{N}[F] \text{ disjoint edges}\}$

Example





Theorem (M - M. R. Pournaki - N. Terai, 2025)

Let $G = G_0[n_1, \ldots, n_h]$ be the multi-whisker graph,
$H = G_0[1, \ldots, 1]$ be the whisker graph.
Set $N_{F_0} := \sum_{j \text{ s.t } y_j \in F_0} n_j + r_x$, where $r_x := |\{k \mid x_k \in F_0\}|$.

Then,
$$F(H^i_{\mathfrak{m}}(\Bbbk[V(G)]/I(G)), t) = \sum_j \dim_{\Bbbk}[H^i_{\mathfrak{m}}(\Bbbk[V(G)]/I(G))]_j t^j$$
$$= \sum_{\substack{F_0 \in P(H) \\ N_{F_0} - |F_0| = i - h}} \left(\frac{t^{-1}}{1 - t^{-1}}\right)^{N_{F_0}}.$$

It is known that the following theorem by Grothendieck.

Theorem (A. Grothendieck)

For a $\Bbbk[x_1, \ldots, x_n]$-module of M,

$$i < \operatorname{depth} M \text{ or } \dim M < i \Longrightarrow H^i_{\mathfrak{m}}(M) = 0,$$

$$i = \operatorname{depth} M \text{ or } \dim M \Longrightarrow H^i_{\mathfrak{m}}(M) \neq 0$$

From the above theorem, we obtain that

Corollary

Let $G = G_0[n_1, \ldots, n_h]$ be the multi-whisker graph. Set $d = n_1 + n_2 + \cdots + n_h$. Then

$$H^i_{\mathfrak{m}}(S/I(G)) \neq 0 \ \ (1 \leq \forall i \leq d) \Longleftrightarrow \exists F \in \mathrm{F}(\Delta(G)) \text{ s.t. } |F| = i.$$

Hence we have

Corollary

Let $G = G_0[n_1, \ldots, n_h]$ be the multi-whisker graph. Then

$$\begin{aligned} \operatorname{depth}(S/I(G)) &= \min\{|F|; F \text{ is a maximal independent set of } G\}. \\ &= |V(G)| - \operatorname{bight}(I(G)) \end{aligned}$$

Definition

Let G be a graph, $M = \{e_1, \ldots, e_s\} \subset E(G)$.

- M is a *matching* of G

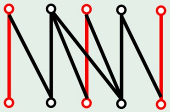
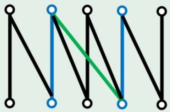
  $\overset{\text{def}}{\iff} \forall e_i, \forall e_j \in M,\ e_i \cap e_j = \phi$

- M is an *induced matching* of G

  $\overset{\text{def}}{\iff}$ M is a matching of G and
  $\forall e \in E(G) \backslash M,\ e \nsubseteq \cup_{i=1}^{s} e_i$

Let $\mathrm{im}(G) := \max\{|M|; M$ is an induced matching of $G\}$
be the *induced matching number* of G

Example

$G =$  $G =$ 

induced matching

not induced matching

$\mathrm{im}(G)=3$

Fact (M. Katzman, 2006)

Let G be a graph. Then,

$$\mathrm{im}(G) \leq \mathrm{reg} S/I(G)$$

It is an important question as to what graphs $\mathrm{im}(G) = \mathrm{reg} S/I(G)$ holds.

Fact

In the following graphs, $\mathrm{im}(G) = \mathrm{reg} S/I(G)$ holds:

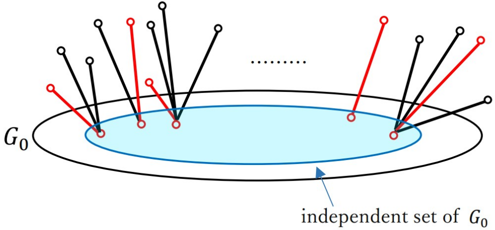
- tree (X. Zheng, 2004)
- chordal graph (H. T. Hà- A. Van Tuyl, 2008)
- unmixed bipartite graph (A. Kummini, 2009)
- sequentially Cohen-Macaulay bipartite graph (A. Van Tuyl, 2009)
- very well-covered graph
  (M. Mahmoudi- A. Mousivand- M. Crupi- G. Rinaldo- N. Terai- S. Yassemi, 2011)
- Cameron-Walker graph (T. Hibi- A. Higashitani- K. Kimura- A. B. O' Keefe, 2015)

Lemma

Let $G = G_0[n_1, \ldots, n_h]$ the multi-whisker graph. Then,

$$\max\{|F_0|;\ F_0 \text{ is an independent set of } G_0\} = \mathrm{im}(G)$$

Indeed,



Corollary

Let $G = G_0[n_1, \ldots, n_h]$ be the multi-whisker graph. Then,

$$\begin{aligned}\mathrm{reg}(S/I(G)) &= \max\{|F_0|;\ F_0 \text{ is an independent set of } G_0\} \\ &= \mathrm{im}(G)\end{aligned}$$

5. The arithmetical rank of edge ideals of multi-whisker graphs

Definition

For a monomial ideal I,

$$\mathrm{ara} I := \min\{r \in \mathbb{N} \ : \ \text{there exist } a_1, \ldots, a_r \in I \text{ such that } \sqrt{(a_1, \ldots, a_r)} = \sqrt{I}\}.$$

the *arithmetical rank* of I

Fact (K. Kimura, 2009)

For a monomial ideal I,

$$I \text{ has a Lyubeznik resolution of length} \ell \Longrightarrow \mathrm{ara} I \leq \ell.$$

Theorem (M - M. R. Pournaki - N. Terai, 2025)

Let $G = G_0[n_1, \ldots, n_h]$ be the multi-whisker graph. Then there exists a order such that the Lyubeznik resolution of $I(G)$ gives a free resolution of the length $\mathrm{bight} I(G)$

Corollary

Let $G = G_0[n_1, \ldots, n_h]$ be the multi-whisker graph. Then,

$$\mathrm{ara}(I(G)) = \mathrm{bight}(I(G))$$

Thank you for your attention!